

**Energy & Power**

Total energy of  $x(t)$ :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$x(t)$  is an **energy signal** iff  $0 < E < \infty$ .

Total power of  $x(t)$ :

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$x(t)$  is a **power signal** iff  $0 < P < \infty$ .

**Remarks.** Energy signals have zero avg power. Power signals have infinite total energy.

**Signal Transformations**

$x(at + b)$ : scale time by  $\frac{1}{|a|}$ , shift by  $-\frac{b}{a}$ .

Energy/power scaling:  $ax(\frac{t}{b})$  gives  $E' = a^2bE$ ,  $P' = a^2P$ .

**Convolution & Unit Impulse**

**Convolution**

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha)y(t - \alpha) d\alpha$$

**Properties:** Commutative, Associative, Distributive.

**Dirac- $\delta$  Function**

$\delta(t) = 0$  for  $t \neq 0$ ;  $\int_{-\epsilon}^{\epsilon} \delta(t) dt = 1$ .

**Properties:**

- Symmetry:  $\delta(t) = \delta(-t)$
- Sampling:  $x(t)\delta(t - \lambda) = x(\lambda)\delta(t - \lambda)$
- Sifting:  $\int_{-\infty}^{\infty} x(t)\delta(t - \lambda) dt = x(\lambda)$
- Replication:  $x(t) * \delta(t - \lambda) = x(t - \lambda)$

**Dirac- $\delta$  Comb**

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Generating function:  $x(t) * \sum_n \delta(t - kn) = \sum_n x(t - kn)$

**Fourier Series**

Any bounded periodic signal  $x_{p(t)}$  with period  $T_p$ :

$$x_{p(t)} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}, \quad f_p = \frac{1}{T_p}$$

**Analysis (coefficients):**

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x_{p(t)} e^{-j2\pi k \frac{t}{T_p}} dt$$

**Trigonometric form:**

$$x_{p(t)} = a_0 + 2 \sum_{k=1}^{\infty} \left[ a_k \cos\left(2\pi k \frac{t}{T_p}\right) + b_k \sin\left(2\pi k \frac{t}{T_p}\right) \right]$$

where  $a_k = \frac{c_{-k} + c_k}{2}$ ,  $b_k = \frac{c_{-k} - c_k}{j2}$ .

**Spectral Symmetry (Fourier Series)**

Signal type	Coefficient properties
Real	$c_k^* = c_{-k}$ $ c_k  =  c_{-k} $ $\angle c_k = -\angle c_{-k}$
Real & even	$c_k^* = c_k$ (real) $c_k = c_{-k}$ (even)
Real & odd	$c_k^* = -c_k$ (imaginary) $-c_k = c_{-k}$ (odd)

**Fourier Transform**

**Dirichlet Conditions**

1. Finite maxima/minima in any finite interval.
2. Finite discontinuities in any finite interval.
3. Absolutely integrable:  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ .

Condition 3 is violated by power signals; (1) and (2) are satisfied by all real-world signals.

**Spectral Symmetry (FT)**

Signal	Spectrum properties
Real $x^*(t) = x(t)$	$X^*(f) = X(-f)$ (conj. symm.) $ X(f)  =  X(-f) $ (even) $\angle X(f) = -\angle X(-f)$ (odd)
Real & even $x^*(t) = x(t)$ , $x(-t) = x(t)$	$X^*(f) = X(f)$ (real) $X(f) = X(-f)$ (even)
Real & odd $x^*(t) = x(t)$ , $x(-t) = -x(t)$	$X^*(f) = -X(f)$ (imaginary) $-X(f) = X(-f)$ (odd)

**Periodic Signals via FT**

If  $x_{p(t)}$  has Fourier series coefficients  $c_k$ :

$$X_{p(f)} = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

Dirac- $\delta$  comb spectrum:  $\text{COMB}_{\lambda(f)} = \frac{1}{\lambda} \sum_k \delta(f - \frac{k}{\lambda})$

**Spectral Density & Bandwidth**

**Energy Spectral Density**

Parseval/Rayleigh:  $E = \int |x(t)|^2 dt = \int |X(f)|^2 df$

$$E_{x(f)} = |X(f)|^2$$

Properties: real,  $\geq 0$ , even if  $x(t)$  real.

**Power Spectral Density**

$$P_{x(f)} = \lim_{W \rightarrow \infty} \frac{1}{2W} |X_{W(f)}|^2, \quad P = \int P_{x(f)} df$$

For periodic  $x_{p(t)}$ :  $P_{x(f)} = \sum_k |c_k|^2 \delta(f - kf_p)$ ,  $P = \sum_k |c_k|^2$

**Bandwidth Definitions**

**Lowpass signal:**  $|X(f)| = 0$  for  $|f| > B$ .

**Bandpass signal:**  $|X(f)| = 0$  for  $\|f| - f_c| > \frac{B}{2}$ .

**3 dB BW (LP):** smallest  $f_B$  where  $|X(f_B)| = |X(0)|/2^{0.5}$  ( $-3.01$  dB).

**3 dB BW (BP):** smallest  $f_B$  where  $|X(f_B)| = |X(f_c)|/2^{0.5}$ ;  $B = f_u - f_l$ .

**1st-null BW (LP):** smallest  $f$  where  $|X(f)| = 0$ .

**1st-null BW (BP):** smallest  $f_u - f_l$  around  $f_c$  where  $|X(f)| = 0$ .

**M% Energy BW (LP):** smallest  $B$  s.t.  $\int_{-B}^B E_{x(f)} df \geq (\frac{M}{100})E$

**M% Energy BW (BP):** smallest  $B$  s.t.  $\int_{f_c - \frac{B}{2}}^{f_c + \frac{B}{2}} E_{x(f)} df \geq (\frac{M}{100})(\frac{E}{2})$

**M% Power BW (periodic):**  $B = Kf_p$  where  $K$  is smallest integer s.t.

$$\sum_{k=-K}^K |c_k|^2 \geq \frac{M}{100} \times P$$

**Systems — Classification**

**System Types**

Property	Condition
Memoryless	Output depends only on input at same $t$
Causal	Output depends only on present/past input
BIBO Stable	Every bounded input $\Rightarrow$ bounded output
Linear	Superposition holds (additivity + homogeneity)
Time-Invariant	$x(t - \tau) \Rightarrow y(t - \tau)$ for all $\tau$

**Linearity (Superposition)**

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Zero input  $\Rightarrow$  zero output (set all  $\alpha_i = 0$ ).

**Laplace Transform (Definition)**

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt, \quad s = \sigma + j\omega$$

**Partial Fractions — Method**

**Step 1.** If  $\deg(N) \geq \deg(D)$ , perform polynomial long division first.

**Simple real roots**  $F(s) = \frac{N(s)}{(s+a)(s+b)}$ :

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}, \quad A = [(s+a)F(s)]_{s=-a}$$

**Repeated root**  $(s+a)^r$ :

$$\dots + \frac{A_1}{s+a} + \frac{A_2}{(s+a)^2} + \dots + \frac{A_r}{(s+a)^r}$$

$$A_k = \frac{1}{(r-k)!} \left[ \frac{d^{r-k}}{ds^{r-k}} s^{r-k} (s+a)^r F(s) \right]_{s=-a}$$

**Complex conjugate roots**  $s^2 + 2\alpha s + \omega_n^2$ : write as

$$\dots + \frac{As+B}{s^2+2\alpha s+\omega_n^2}$$

Complete the square:  $(s + \alpha)^2 + \omega_d^2$ , then match to damped sin/cos pairs.

**Solving DEs:** Apply  $\mathcal{L} \rightarrow$  rearrange for  $Y(s) \rightarrow$  partial fractions  $\rightarrow$  apply  $\mathcal{L}^{-1}$ .

**LTI Systems**

**Representations**

Domain	Key function
Time	Impulse response $h(t)$
Frequency	Freq. response $H(f)$ or $H(j\omega)$
$s$ -domain	Transfer function $H(s)$

**Convolution (Output)**

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau$$

**Transfer Function**

$$H(s) = \mathcal{L}\{h(t)\} = \frac{Y(s)}{X(s)}$$

(assumes zero initial conditions)

**Frequency Response**

$$H(f) = \mathcal{F}\{h(t)\}, \quad H(j\omega) = H(s)|_{s=j\omega}$$

For causal LTI:  $H(f) = H(j\omega)|_{\omega=2\pi f}$ .

Polar form:  $H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$

**Sinusoidal Steady-State**

Input  $A \cos(\omega_0 t + \psi) \Rightarrow$  output:

$$y(t) = A|H(j\omega_0)| \cos(\omega_0 t + \psi + \angle H(j\omega_0))$$

Amplitude scaled by  $|H(j\omega_0)|$ ; phase shifted by  $\angle H(j\omega_0)$ . Same rule applies for sin.

**General LTI Differential Equation**

$$\sum_{n=0}^N a_n \frac{d^n y}{dt^n} = \sum_{m=0}^M b_m \frac{d^m x}{dt^m}$$

$$H(s) = \frac{b_M s^M + \dots + b_0}{a_N s^N + \dots + a_0}$$

**Poles**  $-p_n$ : roots of denominator;  $H(-p_n) = \infty$ .

**Zeros**  $-z_m$ : roots of numerator;  $H(-z_m) = 0$ .

**Pole-zero excess** =  $N - M$ .

**System Stability (from Poles)**

Type	Pole location	$h(t)$ as $t \rightarrow \infty$
BIBO Stable	All in LHP ( $\text{Re}\{s\} < 0$ )	$\rightarrow 0$
Marginally stable	Simple poles on $j\omega$ -axis	Bounded $\neq 0$
Unstable (I)	Any pole in RHP ( $\text{Re}\{s\} > 0$ )	$\rightarrow \infty$
Unstable (II)	Repeated poles on $j\omega$ -axis	$\rightarrow \infty$

Zeros do NOT affect stability — poles only.

**Step Response**

$$o(t) = \int_{-\infty}^t h(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$$

### First-Order System

$$T \frac{dy}{dt} + y(t) = Kx(t) \rightarrow H(s) = \frac{K}{Ts+1}$$

- Pole:  $s = -\frac{1}{T}$ ; Time constant:  $T$
- Impulse resp.:  $h(t) = \left(\frac{K}{T}\right)e^{-\frac{t}{T}}u(t)$
- Step resp.:  $o(t) = K(1 - e^{-\frac{t}{T}})u(t)$ ; reaches 63.2% of  $K$  at  $t = T$

### Second-Order System

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{Poles: } s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta$	Type	Poles
$> 1$	Overdamped	Real, distinct
$= 1$	Critically damped	Real, repeated: $-\omega_n$
$0 < \zeta < 1$	Underdamped	Complex conjugate
$= 0$	Undamped	Pure imaginary $\pm j\omega_n$

**Underdamped** ( $0 < \zeta < 1$ ):  $\sigma = \zeta\omega_n$ ,  $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

$$\text{Impulse resp.: } h(t) = \left(K\frac{\omega_n^2}{\omega_d}\right)e^{-\sigma t} \sin(\omega_d t)u(t)$$

$$\text{Step resp.: } o(t) = K\left[1 - e^{-\sigma t} \left(\cos\omega_d t + \left(\frac{\sigma}{\omega_d}\right) \sin\omega_d t\right)\right]u(t)$$

**Critically damped** ( $\zeta = 1$ ):

$$\text{Impulse resp.: } h(t) = K\omega_n^2 t e^{-\omega_n t} u(t)$$

$$\text{Step resp.: } o(t) = K[1 - e^{-\omega_n t}(1 + \omega_n t)]u(t)$$

**Identifying parameters** from  $H(s) = \frac{K\omega_n^2}{s^2 + b_1s + b_0}$ :

- $\omega_n = \sqrt{b_0}$
- $\zeta = \frac{b_1}{2\omega_n}$
- DC gain =  $K$

### Bode Diagrams

Bode plot = straight-line approx. of  $H(j\omega)$  on semilog- $\omega$  axes.

- **Magnitude:**  $|H(j\omega)|_{\text{dB}} = 20 \log_{10}|H(j\omega)|$
- **Phase:**  $\angle H(j\omega)$  in degrees

### Standard Form

$$H(s) = K_{\text{dc}} \cdot \prod_m \left(1 + \frac{s}{z_m}\right) \cdot \prod_q \frac{\omega_{n,q}^2}{s^2 + 2\zeta_q \omega_{n,q} s + \omega_{n,q}^2}$$

where  $K_{\text{dc}} = H(0)$  (DC gain).

With a differentiator:  $K_d \cdot s \cdot \dots$ ; integrator:  $\left(\frac{K_i}{s}\right) \cdot \dots$

### Building Blocks

Block	Mag slope	Phase
DC gain $K_{\text{dc}}$	0 dB/dec	0°
Differentiator $K_d s$	+20 dB/dec	+90°
Integrator $\frac{K_i}{s}$	-20 dB/dec	-90°
Zero $\left(1 + \frac{s}{z_m}\right)$	+20 after $z_m$	0° → +90°
Pole $\frac{1}{1 + \frac{s}{p_n}}$	-20 after $p_n$	0° → -90°
2nd-order factor	-40 after $\omega_n$	0° → -180°

Phase transition spans one decade either side of the corner frequency ( $\pm 45^\circ$ /dec).

### Asymptotic Rules

$N$  = no. of poles,  $M$  = no. of zeros.

**HF phase:**  $(N - M) \times -90^\circ$

**LF phase:**  $(\text{No. of } \int dt - \text{No. of } \frac{d}{dt}) \times -90^\circ$

**HF slope:**  $(N - M) \times -20$  dB

**LF slope:**  $(\text{No. of } \int dt - \text{No. of } \frac{d}{dt}) \times -20$  dB

### Reading Transfer Function from Bode Plot

1. **Low-freq slope:** +20L dB/dec  $\Rightarrow L$  differentiators;  $-20L \Rightarrow L$  integrators;  $0 \Rightarrow$  DC gain  $K_{\text{dc}}$ .
2. Read DC gain from low-freq intercept.
3. Each **slope change of +20** dB/dec  $\Rightarrow$  zero at that corner freq.
4. Each **slope change of -20** dB/dec  $\Rightarrow$  pole.
5. **Slope change of -40** dB/dec  $\Rightarrow$  2nd-order factor (check for resonance hump).
6. Assemble into  $H(s) = \frac{K(1 + \frac{s}{z_1})}{\left(1 + \frac{s}{p_1}\right) \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$  form.

### Resonance (2nd-Order)

Valid only for  $\zeta < \frac{1}{\sqrt{2}} \approx 0.707$  (i.e.  $\zeta^2 < \frac{1}{2}$ ):

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}, \quad |H(j\omega_r)|_{\text{peak}} = \frac{K}{2\zeta\sqrt{1 - \zeta^2}}$$

When  $\zeta \rightarrow 0$ :  $\omega_r \rightarrow \omega_n$ , peak  $\rightarrow \infty$ .

### Sampling & Reconstruction

#### Ideal Filters

**Ideal LPF** (cutoff  $B$  Hz, gain  $A$ ):

$$H(f) = A \cdot \text{rect}\left(\frac{f}{2B}\right), \quad h(t) = 2AB \text{sinc}(2Bt)$$

**Ideal BPF** (centre  $f_o$ , BW  $B = f_u - f_l$ ):

$$h(t) = 2AB \text{sinc}(Bt) \cos(2\pi f_o t)$$

Both are **non-causal** and physically unrealizable; approximated by Butterworth, Chebyshev, etc.

#### Sampling Process

Multiply  $x(t)$  by impulse train with period  $T_s = \frac{1}{f_s}$ :

$$x_{s(t)} = x(t) \cdot \sum_n \delta(t - nT_s)$$

Spectrum (replicated at multiples of  $f_s$ ):

$$X_{s(f)} = f_s \sum_{k=-\infty}^{\infty} X(f - kf_s)$$

#### Nyquist Sampling Theorem

For a signal band-limited to  $f_m$  Hz:

$$f_s \geq 2f_m \quad (\text{Nyquist rate} = 2f_m)$$

- $f_s \geq 2f_m$ : **no aliasing**, perfect reconstruction possible.
- $f_s < 2f_m$ : **aliasing** — spectral images overlap, reconstruction impossible.

#### Reconstruction Filter

Use ideal LPF with:

$$K = \frac{1}{f_s} = T_s, \quad f_m \leq f_c \leq f_s - f_m$$

Best choice:  $f_c = f_m$  (minimum bandwidth). Output:  $\hat{x}(t) = x(t)$ .

#### Bandpass Sampling (below Nyquist rate)

Bandpass signal: centre  $f_c$ , bandwidth  $B$ . Full Nyquist rate =  $2(f_c + \frac{B}{2})$ .

**Overlapping images** — condition 9.2a (Symmetric):

$$f_s = \frac{2f_c}{k}, \quad k = 1, 2, \dots, \left\lfloor \frac{2f_c}{B} \right\rfloor, \quad f_{s,\text{min}} = \frac{2f_c}{\lceil 2f_c/B \rceil}$$

**Un-aliased images** — condition 9.2b (Asymmetric):

$$f_s \in \left[ \frac{2f_c + B}{k+1}, \frac{2f_c - B}{k} \right], \quad k = 1, 2, \dots, \left\lfloor \frac{2f_c - B}{2B} \right\rfloor, \quad f_{s,\text{min}} = \frac{2f_c + B}{\lceil (2f_c - B)/(2B) \rceil + 1}$$

Minimum sub-Nyquist rate  $\approx 2B$  (equal to signal bandwidth).

**Reconstruction filter:** Ideal BPF covering the signal band (not LPF).

#### Sampling Decision Checklist

1. Is  $x(t)$  band-limited to  $f_m$ ? (required for perfect reconstruction.)
2. Is  $f_s \geq 2f_m$ ? Yes  $\Rightarrow$  ideal LPF with  $K = T_s$ , cutoff  $f_m \leq f_c \leq f_s - f_m$ .
3. Bandpass signal? Check (9.2a) or (9.2b) for valid sub-Nyquist  $f_s$ .
4. Reconstruction filter: LPF if sampled at/above Nyquist; BPF for bandpass sub-Nyquist.

### Quick Solve Methods

#### Solving DEs (Laplace Method)

1. Apply  $\mathcal{L}$  with zero ICs:  $\frac{d^n y}{dt^n} \leftrightarrow s^n Y(s)$ .
2. Rearrange:  $Y(s) = H(s) \cdot X(s)$ .
3. Expand via partial fractions.
4. Apply  $\mathcal{L}^{-1}$  using table.

#### Finding Steady-State Output

1. Identify  $\omega_0$  from input.
2. Compute  $H(j\omega_0)$ : find magnitude and phase.
3. Scale and shift:  $y(t) = A|H(j\omega_0)| \cos(\omega_0 t + \psi + \angle H(j\omega_0))$ .

#### Stability Check

- Compute poles (roots of denominator of  $H(s)$ ).
- All poles in LHP  $\Rightarrow$  BIBO stable.
- Any pole in RHP  $\Rightarrow$  unstable.
- Simple poles on  $j\omega$ -axis  $\Rightarrow$  marginally stable.
- Repeated poles on  $j\omega$ -axis  $\Rightarrow$  unstable.

#### Bode Plot (Step-by-Step)

1. Rewrite  $H(s)$  in standard form (factored, unity-coefficient building blocks).
2. Compute  $K_{\text{dc}}$  (or  $K_d/K_i$ ) and starting level in dB.
3. List corner frequencies in ascending order.
4. Draw low-freq asymptote; add slope changes at each corner.
5. Phase: each pole/zero contributes  $\pm 45^\circ$ /dec over one decade either side of its corner.
6. Sum all contributions (dB magnitudes add linearly).

#### Complex Number Polar Form

$$a + jb = \sqrt{a^2 + b^2} \cdot e^{j \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\frac{1}{c + jd} = \frac{1}{\sqrt{c^2 + d^2}} \cdot \exp\left(-j \tan^{-1}\left(\frac{d}{c}\right)\right)$$

#### Angle Correction Quick-Reference

For  $H(j\omega) = \frac{a + jb}{\dots}$ , after computing  $\theta = \arctan\left(\frac{b}{a}\right)$ :

- $a > 0$ : use  $\theta$  as-is
- $a < 0, b > 0$ : add 180° (2nd quadrant)
- $a < 0, b < 0$ : subtract 180° (3rd quadrant)